A THIRD GENERALIZATION OF THE GROUPS OF THE REGULAR POLYHEDRONS.

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§ 1. Introduction.

On November 10, 1856, Sir William R. Hamilton presented before the Irish Academy a paper entitled "A new system of roots of unity" * in which he pointed out the interesting fact that if s_1, s_2 represent two operators, which obey the associative but not the commutative law of multiplication, the three sets of three equations,

$$s_1^2 = s_2^3 = (s_1 s_2)^r = 1; \qquad r = 3,\ 4,\ 5,$$

define the groups of movements of the regular polyhedrons. The case when $r = 5$ seems to have interested him especially and he denoted the resulting group in this special case by "Icosian Calculus," observing that all these results may be represented geometrically on the regular icosahedron or on the regular dodecahedron. About a quarter of a century later Dyck rediscovered the same results and put the whole matter in a somewhat clearer form from the standpoint of abstract groups.† These relations are so simple and admit such a variety of geometric interpretations that they have become classic.

Two generalizations of these Hamiltonian relations were developed about a quarter of a century after the publications by Dyck, or about half a century after Hamilton had started investigations in this direction. In the earlier of these generalizations ‡ the groups generated by s_1, s_2 when two of the three Hamiltonian relations are replaced by a single one, without changing the third, were investigated, and the possible groups were determined. It was assumed throughout this article that the generating operators were not commutative. The special cases when these operators are commutative were considered incidentally in a later article.§ In the latter article a few errors relating to the first generalizations of the icosahedral group were also corrected.

A second generalization of the given Hamiltonian relations appeared

* Hamilton, Proceedings of the Royal Irish Academy, vol. 6 (1853–7), p. 415.

† Dyck, Mathematische Annalen, vol. 22 (1883), p. 82.

‡ Transactions of the American Mathematical Society, vol. 8 (1907), p. 1.

§ Quarterly Journal of Mathematics, vol. 41 (1910), p. 171.

in two articles. The former of these was devoted to a second generalization of the relations defining the tetrahedral and the octahedral groups * while the latter confined itself to a second generalization of the icosahedral group. † In these second generalizations the third Hamiltonian relation $(s_1s_2)^r = 1$ was replaced by $(s_1s_2)^r = (s_2s_1)^r$, while the other two relat ons were replaced by a single one, just as in the first generalizations. While only a small number of different groups involve two generators which satisfy the former of these generalized conditions, there is always an infinite number of such groups when the latter generalized relations are assumed to be the only conditions impoted on the two generators.

The generalizations considered in the present paper are more direct than those previously considered as they are obtained from the Hamiltonian relations by omitting the condition that the given powers of s_1, s_2, s_1s_2 are equal to unity. That is, we consider relations of the general type

$$s_1^2 = s_2^3 = (s_1s_2)^r, \qquad r = 3,\ 4,\ 5.$$

Such relations are evidently equivalent to the two conditions $s_1^2 = s_2^3$, $s_1^2 = (s_1s_2)^r$ and it will be proved that they are satisfied by the two generators of at least two and at most four groups for each value of r. The simplicity of these results seems to justify the hope that they may find extensive applications.

It should perhaps be emphasized that the results here obtained are considerably simpler than those obtained from the other two generalizations that have been noted, and hence the present paper has closer contact with the original developments by Hamilton than these earlier generalizations have. The reason that the present generalizations were not developed first is that the writer did not foresee that they would lead to simple results, and did not notice an easy approach to this problem when the other generalizations were taken up.

The method employed in § 2 is quite different from that used in the following sections. This change is due to the fact that it was thought that the former method would give a deeper insight into the problem and the considerations of § 2 were sufficiently simple to employ a more general method of work than seemed feasible in the other sections where the considerations become more complex. It may however be well to indicate here how to prove the main results of § 2 by the more special methods employed later.

Starting with the equations $s_1^2 = s_2^3 = (s_1s_2)^3$ we observe at once that s_1, $s_2^{-1}s_1s_2$ have a common square, since s_1^2 is invariant under the group G generated by s_1, s_2. Hence $s_1s_2^{-1}s_1^{-1}s_2$ is transformed into its inverse by s_1

* American Journal of Mathematics, vol. 32 (1910), p. 65.

† Quarterly Journal of Mathematics, vol. 41 (1910), p. 168.

according to the theorem: If two operators have a common square the product of one and the inverse of the other is transformed into its inverse by each of these operators.* By means of the equations,

$$s_1 = s_2s_1s_2s_1s_2, \qquad s_2^2 = s_1s_2s_1s_2s_1,$$

which are equivalent to the given conditional equations, it is easy to verify the following relations:

$$(s_1s_2^{-1}s_1^{-1}s_2)^2 = (s_2s_1s_2^2)^2 = s_2s_1s_2^3s_1s_2^2 = s_1^6.$$

As s_1^6 is both invariant under s_1 and also transformed into its inverse by s_1, its order divides 12. Hence the order of s_2 must divide 18 and the order of G divides 72.†

After having proved that the order of G divides 72 whenever G is generated by two operators which satisfy the given conditions, it is comparatively easy to determine the total number of groups which may be generated by two such operators, and these details are given in the following section. In a similar way it may be observed that when

$$s_1^3 = s_2^3 = (s_1s_2)^2; \text{ or } s_1^2 = s_2s_1s_2, \; s_2^2 = s_1s_2s_1$$

the two operators s_1s_2, s_2s_1 have a common square, and hence

$$(s_1s_2s_1^{-1}s_2^{-1})^2 = s_1^{-6}(s_1s_2s_1^2s_2^{-1})^2 = s_1^{-6}(s_1s_2^2s_1)^2 = s_1^{-6}(s_1^2s_2s_1^2)^2$$
$$= s_1^{-3}s_1^2s_2s_1s_2s_1^2 = s_1^3.$$

Since s_1^3 is invariant under s_1s_2 and also transformed into its inverse by s_1s_2, the order of s_1 divides 6 and the order of G divides 24. Hence it is again very easy to complete the determination of all the possible groups which can be generated by two operators satisfying the given conditions. These special developments appear in the following section.

It may be added that if a set of conditions in the form of equations is given this set is always satisfied by the assumption that each of the operators is the identity. This trivial special case is not generally mentioned in what follows as it is so evident and exists always. Hence it must generally be assumed in the following theorems that at least one of the operators under consideration is not the identity. This is so often done in group theory literature that it scarcely calls for justification in an article. The types of alternative proofs suggested for the results of § 2 may also be employed to prove many of the results of the following sections.

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* Archiv der Mathematik und Physik, vol. 9 (1905), p. 6.

† This upper limit of the order of G is a direct result of the well known theorem that two non-commutative operators s_1, s_2 which satisfy the equations $s_1^2 = s_2^3 = (s_1s_2)^3 = 1$ must generate the tetrahedral group.

§ 2. Generalization of the tetrahedral group.

If the two non-commutative generators s_1, s_2 of a group G satisfy the conditions

$$s_1^2 = s_2^3 = (s_1 s_2)^3$$

it results directly that the cyclic group generated by s_1^2 is the central of G, and that the quotient group of G with respect to its central (the central quotient group of G) is the tetrahedral group. To the invariant subgroup of order 4 in this central quotient group there corresponds a subgroup of G whose operators of odd order are in its central and hence this subgroup is the direct product of a cyclic group of odd order and a group of order 2^α. We proceed to prove that the latter is either the four group or the quaternion group. In fact, this follows directly from the necessary property of this group that it involves three cyclic subgroups of order $2^{\alpha-1}$ which are conjugate under G. It is well known that there are only two groups of order 2^α which contain three cyclic subgroups of order $2^{\alpha-1}$ and that the groups of order 2^α which contain more than one cyclic subgroup of order $2^{\alpha-1}$ are conformal with abelian groups whenever $\alpha > 3$.

From the preceding paragraph it follows directly that the order of G cannot be divisible 16 whenever G is generated by s_1, s_2 subject to the given conditions. It is also easy to see that G cannot involve a subgroup of half its own order, since such a subgroup would involve exactly half the operators of G which correspond to each operator in its central quotient group. In particular, this subgroup would involve half of the central of G, but this half central could not involve the square of any operator corresponding to an operator of order 2 in the central quotient group and hence G cannot involve a subgroup of half its own order.

As there is only one group of order 24 which does not contain a subgroup of order 12, it results from the preceding paragraph that G must be this non-twelve group of order 24 when the order of s_1 is 4. From the properties of this group of order 24 it is clear that it can be generated by two operators of orders 4 and 6 respectively, which satisfy the conditions imposed on s_1 and s_2. That is, when s_1 is of order 2, G is the tetrahedral group, and when s_1 is of order 4, G is the non-twelve group of order 24. We shall soon be able to prove that the order of G must divide 72 and that the two non-commutative operators s_1, s_2 must generate one of four groups when they satisfy the given conditions.

To establish this fact, it is convenient to make use of a theorem which has very wide applications and may be stated as follows: *If $s_1, s_2, \cdots, s_\rho$ is a complete set of conjugates, in order, under an operator t and if the continued product of this set of conjugates, in order, is the identity then will* $(s_\alpha t)^\rho = t^\rho$,

where α is any one of the numbers 1, 2, $\cdots$, ρ. Since $t^{-1}s_\alpha t = s_{\alpha+1}(\alpha = 1, 2, \cdots, \rho - 1)$ it results that $s_\alpha t = ts_{\alpha+1}$. Hence $(s_\alpha t)^\rho = s_\alpha s_{\alpha-1} \cdots s_{\alpha+1}t^\rho$, where $\alpha - 1$, $\alpha - 2$, $\cdots$ are to be replaced by their least positive residues modulo ρ except that 0 is replaced by ρ. From the fact that $s_1 s_2 \cdots s_\rho = 1$ it results that $s_\alpha s_{\alpha-1} \cdots s_{\alpha+1} = 1$ and hence the theorem is proved. It is clear that the given theorem remains true when the set $s_1, s_2, \cdots, s_\rho$ represents more than one complete set of conjugates, in order, under t, since the given proof does not depend upon the fact that the operators $s_1, s_2, \cdots, s_\rho$ are distinct. Other generalizations of the given theorem at once suggest themselves but for our present purpose it is convenient to leave it in the special form in which it has been stated.

It has been observed above that s_1 may be regarded as the direct product of $s_2^{3\alpha}$ and an operater s_1' which is either of order 2 or of order 4, and that s_2 transforms s_1' into three conjugates s_1', s_2', s_3' whose continued product is the identity when s_1' is of order 2, since these three operators of order 2 and the identity constitute the four group. This continued product must also be the identity when s_1' is of order 4 since $(s_1's_2)^3$ is of even order in this case. Hence we have that $s_1'\, s_2'\, s_3' = 1$ in all cases, and if we combine this with the theorem of the preceding paragraph it results that the conditions given at the beginning of this section may be replaced by

$$s_1'^2 s_2^{6\alpha} = s_2^3 = (s_1's_2)^3 s_2^{9\alpha} = s_2^{9\alpha+3}.$$

Hence $s_2^{9\alpha} = 1$, and as α and the order of s_2 have only 2 or 1 as their highest common factor it results that $s_2 = 1^{18}$.

To prove that the order of s_2 may be 18 it may be convenient to begin with the case when s_1, s_2 are commutative and hence the given conditions reduce to

$$s_1^2 = s_2^3 = s_1^3 s_2^3.$$

From these conditions it results directly that $s_1^3 = 1$, and hence $s_2^9 = 1$. Moreover, if s_2 is an operator of order 9 and $s_1 = s_2^6$, it is evident that s_1, s_2 satisfy the given conditions and generate the cyclic group of order 9. As it has been observed that s_1, s_2 may be so chosen that their orders are 4 and 6 respectively and that they generate the non-twelve group of order 24, it results directly that we may associate with these two non-commutative operators two commutative ones t_1, t_2 (which are also commutative with s_1, s_2) of orders 3 and 9 respectively so that s_1t_1, s_2t_2 are two operators of order 12 and 18 respectively which satisfy the given conditions. In the same way we see that the orders of s_1, s_2 may be 6 and 9 respectively and hence there results the theorem: *If two non-commutative operators s_1, s_2 satisfy the two conditions $s_1^2 = s_2^3 = (s_1s_2)^3$ they must generate one of the following four groups: the tetrahedral group, the non-twelve group of order* 24,

or the groups obtained by establishing a tris-isomorphism between each of these groups and the cyclic group of order* 9. *When* s_1, s_2 *are commutative and satisfy these conditions they generate either the cyclic group of order* 9 *or the group of order* 3.

Closely related to the generalization considered above is the following:

$$s_1^3 = s_2^3 = (s_1 s_2)^2.$$

That this does not lead to the same category as the set considered above is evident from the fact that if s_1, s_2 are commutative and satisfy these conditions they generate the group of order 3, while in the preceding case they could also generate the group of order 9. We shall again begin with the case when these generating operators are non-commutative, and hence G is a group whose central quotient group is the tetrahedral group. Just as in the preceding case we observe that, when s_1s_2 is of order 2 or 4, G is the tetrahedral group or the non-twelve group of order 24, and that G cannot involve a subgroup of index 2. We proceed to prove that G must be one of these two groups when s_1, s_2 are non-commutative and satisfy these conditions.

The subgroup of G which corresponds to the four-group in the central quotient group is the direct product of a group of odd order and either the four-group or the quaternion group, for the same reasons as were given under the preceding conditions. Hence we may assume that $s_1s_2 = s's_1^{3\alpha}$, where s' is either of order 2 or 4 and is commutative with $s_1^{3\alpha}$. Moreover, α has at most the factor 2 in common with the order of s_1 since $s_1^{3\alpha}$ must generate the group of odd order in the central of G. Hence the following equations:

$$(s_1s_2)^2 = s'^2 s_1^{6\alpha} = s_1^3; \quad s_1^{12\alpha} = s_1^6.$$

On the other hand, we have the equations

$$s_1^{-1}\cdot s_1s_2 = s_1^{-1}s'\cdot s_1^{3\alpha}; \quad s_2^6 = s_1^6 = s_1^{-6}s_1^{18\alpha},$$

in accord with the general theorem given above. From these equations it results directly that

$$s_1^{24\alpha} = s_1^{18\alpha}, \text{ or } s_1^{6\alpha} = 1.$$

As the order of s_1 cannot be divisible by 4 it results that the order of s_1 divided 6, and hence the order of s_2 is also a divisor of this number. This

* A tris-isomorphism is one in which the invariant subgroups of index 3 are made to correspond, hence the order of the resulting group is one-third of the product of the orders of the isomorphic groups. When the invariant subgroups of index 2 are made to correspond the isomorphism may be called a dim-isomorphism. Cf. Cayley, Quarterly Journal of Mathematics, vol. 25 (1890), p. 85.

proves that the order of s_1s_2 is either 2 and 4 and completes the proof of the theorem: *If two non-commutative operators satisfy the conditions* $s_1^3 = s_2^3 = (s_1s_2)^2$ *they generate either the tetrahedral group or the non-twelve group of order* 24. *If two commutative operators satsify these conditions they generate the group of order* 3 *unless each of these operators is the identity.*

In speaking of the finite group generated by operators subject to certain cond tions it is customary to think of the largest possible group which these operators can generate under these conditions. If this were done in the present section the two theorems on the generalization of the tetrahedral group could be expressed as follows: If two operators satisfy the conditions $s_1^3 = s_2^3 = (s_1s_2)^2$ they generate the non-twelve group of order 24, and if they satisfy the conditions $s_1^2 = s_2^3 = (s_1s_2)^3$ they generate the group of order 72 formed by establishing a tris-isomorphism between this group of order 24 and the cyclic group of order 9. The theorems as stated above are, however, more definite as they include the cases when s_1, s_2 do not generate the largest possible group subject to the given conditions.

§ 3. Generalizations of the octahedral group.

If the two non-commutative operators s_1, s_2 satisfy the two conditions

$$s_1^3 = s_2^4 = (s_1s_2)^2,$$

the two operators s_2, $s_1^{-1}s_2^2s_1$ have a common square, and hence the product of one of these into the inverse of the other is transformed into its inverse by each of these operators. We may therefore endeavor to find an upper limit of the order of the group G generated by s_1, s_2 by finding a power of this product such that this power is invariant under G. From the fact that this power is both transformed into its inverse and is also invariant under the same operator, it results directly that its order cannot exceed 2. The actual work may be performed as follows.

From the given relations it results directly that

$$s_1s_2s_1 = s_2^3 \quad \text{and} \quad s_2s_1s_2 = s_1^2.$$

Hence it is not difficult to derive the following relations:

$$\begin{aligned}(s_2^2s_1^{-1}s_2^{-2}s_1)^2 &= s_2^2s_1^{-1}\cdot s_2^{-2}s_1s_2^2\cdot s_1^{-1}s_2^{-2}s_1 = s_2^2s_1^{-1}s_2^{-3}s_1^2s_2s_1^{-1}s_2^{-2}s_1\\ &= s_2^2s_1^{-2}s_2^{-1}s_1s_2s_1^{-1}s_2^{-2}s_1 = s_2^2s_1^{-2}s_2^{-2}s_1s_2^{-2}s_1\\ &= s_2^2s_1^{-2}s_2^{-3}s_1^2s_2^{-3}s_1 = s_1^{-9}s_2^2s_1s_2s_1^2s_2s_1\\ &= s_1^{-9}s_2^5s_1s_2s_1 = s_1^{-3}.\end{aligned}$$

This proves that the order of s_1 is a divisor of 6 whenever the given con-

ditions are satisfied, and hence the order of G cannot exceed 48. Moreover, it is easy to see that the non-twelve group of order 24 can be extended by means of an operator of order 4 so as to obtain a group of order 48 which may be generated by two operators satisfying the given condition. Hence the theorem: *If two non-commutative operators fulfil the condition* $s_1^3 = s_2^4 = (s_1s_2)^2$ *they generate either the octahedral group or a group of order* 48 *known as* G_{52}.* *If two commutative operators satisfy these conditions they evidently generate the group of order* 2.

If two non-commutative operators satisfy the two conditions

$$s_1^2 = s_2^4 = (s_1s_2)^3; \quad \text{or} \quad s_1 = s_2s_1s_2s_1s_2, \; s_2^3 = s_1s_2s_1s_2s_1,$$

the two operators, s_2^2, $s_1^{-1}s_2^2s_1$ have again a common square and hence we shall consider the reduced form of $(s_2^2s_1^{-1}s_2^{-2}s_1)^2$ as follows

$$\begin{aligned}(s_2^2s_1^{-1}s_2^{-2}s_1)^2 &= s_2^2s_1^{-1}s_2^{-2}s_1s_2^2s_1^{-1}s_2^{-2}s_1 = s_2^2s_1^{-1}s_2^{-1}s_1s_2s_1s_2^3s_1^{-1}s_2^{-2}s_1 \\ &= s_2^2s_1^{-1}s_2^{-1}s_1s_2s_1^2s_2s_1s_2^{-1}s_1 = s_2^2s_1s_2^{-1}s_1s_2^2s_1s_2^{-1}s_1 \\ &= s_2^2s_1^2s_2s_1s_2^3s_1s_2^{-1}s_1 = s_1^2(s_2^3s_1)^3s_2^{-4} = s_1^{10}.\end{aligned}$$

Hence it results that the order of s_1 is a divisor of 20 and that the order of G divides 240. If s_1 is actually of order 20 the order of s_2 must be 40, and s_1^{15}, s_1^5 satisfy the given relations. These operators must therefore generate G_{52}, and G must be the direct product of this G_{52} and the group of order 5. On the other hand, if the order of s_1 is 10 the given conditions are again satisfied and G must be the direct product of the octahedral group and the group of order 5. If s_1, s_2 are commutative and satisfy the given conditions they clearly generate either the group of order 5 or the cyclic group of order 10. Hence the theorem: *If two non-commutative operators satisfy the conditions* $s_1^2 = s_2^4 = (s_1s_2)^3$ *they generate one of the following four groups; the octahedral group,* G_{52}, *or the direct product of one of these groups and the group of order* 5. *When two commutative operators satisfy these conditions, they generate the group of order* 2 *or of order* 5, *or the cyclic group of order* 10.

The third and last generalization of the octahedral group to be considered in this connection is given by the equations

$$s_1^2 = s_2^3 = (s_1s_2)^4.$$

From these equations we easily deduce an equivalent system as follows:

$$s_1 = s_2s_1s_2s_1s_2s_1s_2, \; s_2^2 = s_1s_2s_1s_2s_1s_2s_1.$$

We shall proceed again in the same manner as in the two preceding cases observing that $s_1s_2s_1s_2$ and $s_2s_1s_2s_1$ have a common square since $(s_1s_2)^4 = (s_2s_1)^4$ is invariant under G. Hence the following equations

* Quarterly Journal of Mathematics, vol. 30 (1898–9), p. 258.

$$(s_1s_2s_1s_2s_1^{-1}s_2^{-1}s_1^{-1}s_2^{-1})^2 = s_1^{-6}(s_1s_2s_1s_2s_1s_2^2\cdot s_1s_2^2)^2 = s_1^{-16}(s_2^{-1}s_1^{-1}s_2^4s_1s_2^2)^2$$
$$= s_1^{-20}(s_2^2s_1s_2s_1s_2^4s_1s_2s_1s_2^2) = s_1^{-16}s_2^{-1}(s_1s_2)^4s_2 = s_1^{-14}.$$

Hence the order of s_1 divides 28 and the order of G divides 336.

When s_1, s_2 are commutative it is very easy to see that they generate one of the groups of orders 2 and 7, or the cyclic group of order 14. It is therefore evident that G may be the direct product of G_{52} and the group of order 7. Moreover, when s_1 is of order 28 the two operators s_1^{21}, s_2^7 clearly satisfy the conditions imposed on s_1, s_2 and hence $(s_1^{21}, s_2^7) \equiv G_{52}$. The same operators clearly generate the octahedral group when the order of s_1 is 14. From these results we readily derive the following theorem: *If two non-commutative operators satisfy the conditions $s_1^2 = s_2^3 = (s_1s_2)^4$ they generate one of the following four groups: the octahedral group, the group of order* 48 *known as G_{52}, or the direct product of these groups and the group of order* 7. *If two commutative operators satisfy these conditions they generate one of the following three groups: the groups of orders* 2 *and* 7, *or the cyclic group of order* 14.

It is known that the icosahedral group is generated by s_1, s_2 whenever these two operators satisfy one of the following three sets of three conditions:

$$s_1^3 = s_2^5 = (s_1s_2)^2 = 1, \quad s_1^2 = s_2^5 = (s_1s_2)^3 = 1, \quad s_1^2 = s_2^3 = (s_1s_2)^5 = 1.$$

In the present section we shall consider the possible groups generated by s_1, s_2 when they satisfy one of the following three sets of two conditions:

$$s_1^3 = s_2^5 = (s_1s_2)^2, \quad s_1^2 = s_2^5 = (s_1s_2)^3, \quad s_1^2 = s_2^3 = (s_1s_2)^5.$$

While the number of conditions imposed on s_1, s_2 is thus reduced in each case it will appear that the considerations are not made much more complex thereby. The methods used in the present section are similar to those used in the preceding section and hence they require no further explanations. We shall consider the three cases in the given order.

When $s_1^3 = s_2^5 = (s_1s_2)^2$ it results immediately that

$$s_1^2 = s_2s_1s_2 \quad \text{and} \quad s_2^4 = s_1s_2s_1.$$

Since s_1s_2 and s_2s_1 have a common square we consider the powers of $s_1s_2s_1^{-1}s_2^{-1} = s_1^{-3}s_1s_2s_1^2s_2^{-1} = s_1^{-3}s_1s_2^2s_1$ as follows:

$$(s_1s_2s_1^{-1}s_2^{-1})^2 = s_1^{-6}s_1s_2^2s_1^2s_2^2s_1 = s_1^{-6}s_1s_2s_2^3s_1s_2^3s_1$$

$$(s_1s_2s_1^{-1}s_2^{-1})^3 = s_1^{-9}s_1s_2^3s_1s_2^3s_1^2s_2^2s_1 = s_1^{-9}s_1s_2^3s_1s_2^4s_1s_2^3s_1$$
$$= s_1^{-9}s_1s_2^3s_1^2s_2s_1^2s_2^3s_1 = s_1^{-9}s_1s_2^4s_1s_2^3s_1s_2^4s_1$$
$$= s_1^{-9}s_1^2s_2s_1^2s_2^3s_1^2s_2s_1^2 = s_1^{-9}s_1^2s_2^2s_1s_2^5s_1s_2^2s_1^2$$
$$= s_1^{-6}s_1^2s_2^2s_1^2s_2^2s_1^2$$

$$(s_1s_2s_1^{-1}s_2^{-1})^5 = s_1^{-12}s_1^2s_2^2s_1^2s_2^2s_1^2s_1s_2^3s_1s_2^3s_1 = s_1^3.$$

As s_1^3 is both transformed into its inverse by s_1s_2 and is also invariant under G it results that the order of G divides 120. It is known that there is a group of order 120, known as G_{120}, which is generated by two operators of orders 6 and 10 respectively which satisfy the given conditions,* and hence we have the theorem: *If two operators satisfy the two conditions* $s_1^3 = s_2^5 = (s_1s_2)^2$ *they generate either the icosahedral group or a group of order* 120 *known as* G_{120}. This group of order 120 is the smallest compound perfect group.†

When $s_1^2 = s_2^5 = (s_1s_2)^3$, the two operators s_1, $s_2^{-1}s_1s_2$ have a common square and hence we shall consider the various powers of $s_1s_2^{-1}s_1^{-1}s_2$. These powers may be reduced by means of the equations

$$s_1 = s_2s_1s_2s_1s_2, \quad s_2^4 = s_1s_2s_1s_2s_1$$

which can readily be derived from the given conditions. Hence

$$(s_1s_2^{-1}s_1^{-1}s_2)^2 = s_1^{-4}(s_1s_2^{-1}s_1s_2)^2 = (s_2s_1s_2^2)^2 = s_2s_1s_2^3s_1s_2^2$$

$$(s_1s_2^{-1}s_1^{-1}s_2)^3 = s_1^{-2}s_1s_2^{-1}s_1s_2^2s_1s_2^3s_1s_2^2 = s_2s_1s_2^3s_1s_2^3s_1s_2^2$$

$$= s_2^2s_1s_2s_1s_2^5s_1s_2s_1s_2^5s_1s_2s_1s_2^3 = s_2^2s_1s_2^3s_1s_2^3 \cdot s_1^8$$

$$= s_2^3s_1s_2s_1s_2^5s_1s_2s_1s_2^4 \cdot s_1^8 = s_2^3s_1s_2^2s_1s_2^4 \cdot s_1^{12}$$

$$(s_1s_2^{-1}s_1^{-1}s_2)^5 = s_2s_1s_2^3s_1s_2^2s_2^3s_1s_2^2s_1s_2^4s_1^{12} = s_1^{22}.$$

As s_1^{22} is tranformed into its inverse by s_1 it results that the order of s_1 divides 44 and that the order of G divides 1,320. It is easy to see that if two commutative operators satisfy the given conditions they must have 11 for their common order. Hence the following theorem: *If two non-commutative operators satisfy the two conditions* $s_1^2 = s_2^5 = (s_1s_2)^3$ *they generate one of the following four groups: the icosahedral group,* G_{120} *or the direct product of one of these groups and the group of order* 11. *If two commutative operators satisfy these conditions they generate the group of order* 11.

It remains to consider the case when $s_1^2 = s_2^3 = (s_1s_2)^5$, and hence $s_1 = s_2s_1s_2s_1s_2s_1s_2s_1s_2$, $s_2^2 = s_1s_2s_1s_2s_1s_2s_1s_2s_1$. We shall consider again the powers of $s_1s_2^{-1}s_1^{-1}s_2$, and reduce the expressions for these powers by means of the equations which have just been given. Hence the following equations

$$(s_1s_2^{-1}s_1^{-1}s_2)^2 = s_1^{-4}(s_1s_2^{-1}s_1s_2)^2 = (s_2s_1s_2s_1s_2s_1s_2^2)^2 = s_1^4s_2s_1s_2s_1s_2^2s_1s_2s_1s_2^2$$

$$(s_1s_2^{-1}s_1^{-1}s_2)^3 = s_2s_1s_2s_1s_2s_1s_2^2s_1^4s_2s_1s_2s_1s_2\ s_1s_2s_1s_2^2 = s_1^8s_2s_1s_2s_1s_2^2s_1s_2^2s_1s_2s_1s_2^2$$

$$= s_1^8s_2s_1s_2s_1s_2^2 \cdot s_2s_1s_2s_1s_2s_1s_2s\ s_2 \cdot s_2^2s_1s_2s_1s_2^2$$

$$= s_1^{16}s_2s_1s_2^2s_1s_2s_1s_2^2s_1s_2^2$$

$$(s_1s_2^{-1}s_1^{-1}s_2)^5 = s_1^{20}s_2s_1s_2^2s_1s_2s_1s_2^2s_1s_2^2s_2s_1s_2s_1s_2^2s_1s_2s_1s_2^2 = s_1^{38}.$$

* Transactions of the American Mathematical Society, vol. 8 (1907), p. 10.

† American Journal of Mathematics, vol. 20 (1898), p. 277.

As s_1^{38} is transformed into its inverse by s_1 the order of s_1 is a divisor of 76 and the order of G is a divisor of 2,280. If s_1, s_2 are commutative and satisfy the given conditions they must both be of order 19. Hence it is easy to deduce the following theorem: *If two non-commutative operators satisfy the two conditions* $s_1^2 = s_2^3 = (s_1 s_2)^5$ *they generate one of the following four groups: the icosahedral group,* G_{120} *or the direct product of one of these groups and the group of order* 19. *If two commutative operators satisfy these conditions they enerate the group of order* 19.

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